

Exercise 3.2

1. Let

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$$

Find each of the following:

(i) $A + B$

(ii) $A - B$

(iii) $3A - C$

(iv) AB

(v) BA

Solution:

(i) $A + B$

$$\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$$

(ii) $A - B$

$$\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-3 \\ 3+2 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$$

(iii) $3A - C$

$$3 \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$$

(iv) AB

$$\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2(1)+4(-2) & 2(3)+4(5) \\ 3(1)+2(-2) & 3(3)+2(5) \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$$

(v) BA

$$\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ (-2)(2)+5(3) & (-2)(4)+5(2) \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$$

2. Compute the following:

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$(ii) \begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$$

$$(iii) \begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$$

$$(iv) \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

Solution:

(i)

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$$

(ii)

$$\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix}$$

$$= \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$$

(iv)

$$\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

3. Compute the indicated products:

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad (ii) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 3 \ 4]$$

$$(iii) \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix} \quad (v) \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Solution:

(i)

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a(a)+b(b) & a(-b)+b(-a) \\ -b(a)+a(b) & (-b)(-b)+a(-a) \end{bmatrix}$$
$$= \begin{bmatrix} a^2+b^2 & -2ab \\ 0 & b^2-a^2 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 3 \ 4] = \begin{bmatrix} 1(2) & 1(3) & 1(4) \\ 2(2) & 2(3) & 2(4) \\ 3(2) & 3(3) & 3(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+(-2)2 & 1(2)+(-2)3 & 1(3)+(-2)1 \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1) \end{bmatrix}$$
$$= \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$$

(iv)

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & -12+10+0 & 20+20+30 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$$

(v)

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

(vi)

$$\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ then compute $(A + B)$ and $(B - C)$. Also, verify that $A + (B - C) = (A + B) - C$.

Solution:

Find $A + B$:

$$\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

Find $B - C$:

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0+2 & 3-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

Verify that $A + (B - C) = (A + B) - C$

L.H.S. = $A + (B - C)$

$$\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

R.H.S. = $(A + B) - C$

$$\begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1+2 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

L.H.S. = R.H.S. (Verified).

$$\text{If } A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$$

5.

then compute $3A - 5B$.

Solution:

Find $3A - 5B$:

$$3 \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} - 5 \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Simplify

$$\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$$

Solution:

Simplify first matrix:

$$\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{bmatrix}$$

Simplify second matrix:

$$\sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix}$$

Add results obtained from both the matrices, we get,

$$\begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \end{bmatrix}$$

We know that, $\sin^2 x + \cos^2 x = 1$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This implies,

$$\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Find X and Y, if:

$$(i) \mathbf{X} + \mathbf{Y} = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \text{ and } \mathbf{X} - \mathbf{Y} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(ii) 2\mathbf{X} + 3\mathbf{Y} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \text{ and } 3\mathbf{X} + 2\mathbf{Y} = \begin{bmatrix} -2 & -2 \\ -1 & 5 \end{bmatrix}$$

Solution:

(i) Add both the expressions:

$$(\mathbf{X} + \mathbf{Y}) + (\mathbf{X} - \mathbf{Y}) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$2\mathbf{X} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\mathbf{X} = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Subtract both the expressions:

$$(X + Y) - (X - Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$Y = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(ii) 2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \dots(1) \text{ and}$$

$$3X + 2Y = \begin{bmatrix} -2 & -2 \\ -1 & 5 \end{bmatrix} \dots(2)$$

Multiply equation (1) by 2,

$$4X + 6Y = 2 \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} \dots\dots(3)$$

Multiply equation (2) by 3

$$9X + 6Y = 3 \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & -15 \end{bmatrix} \dots\dots(4)$$

Subtract equation (3) from (4)

$$5X = \begin{bmatrix} -6 & 6 \\ 3 & -15 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ -5 & -15 \end{bmatrix}$$

$$X = \frac{1}{5} \begin{bmatrix} -10 & 0 \\ -5 & -15 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

8. Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

Solution:

$$2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y$$

$$2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

9. Find x and y if

$$2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Solution:

Solving left hand side expression, we get

$$2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Find x and y:

Equate corresponding elements of the matrices:

$$2 + y = 5 \Rightarrow y = 3$$

$$2x + 2 = 8 \Rightarrow x = 3$$

10. Solve the equation for x, y, z and t and if

$$2 \begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3 \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Solution:

$$2 \begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3 \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 2z-3 \\ 2y+0 & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Find x, y, z and t by equating corresponding entries:

$$2x + 3 = 9 \Rightarrow x = 3$$

$$2y = 12 \Rightarrow y = 6$$

$$2z - 3 = 15 \Rightarrow z = 9$$

$$2t + 6 = 18 \Rightarrow t = 6$$

11. If $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ find the values of x and y.

Solution:

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x-y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Find x and y by equating corresponding entries

$$2x - y = 10 \dots(i)$$

$$3x + y = 5 \dots(ii)$$

Add both the equation,

$$5x = 15 \Rightarrow x = 3$$

Put value of x in equation (ii),

$$9 + y = 5 \Rightarrow y = -4$$

12. Given $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w.

Solution:

$$3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

Find x, y, z and w by equating corresponding entries

$$3x = x + 4 \Rightarrow x = 2$$

$$3z = -1 + z + w \dots\dots(1)$$

$$3y = 6 + x + y \dots\dots(2)$$

$$3w = 2w + 3 \Rightarrow w = 3$$

Put value of w in equation (1)

$$3z = -1 + z + 3 \Rightarrow 2z = 2 \Rightarrow z = 1$$

Put value of x in equation (2)

$$3y = 6 + 2 + y \Rightarrow y = 4.$$

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x) F(y) = F(x + y)$.

Solution:

Change x to y

$$F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(x) F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y + 0 & -\cos x \sin y - \sin x \cos y + 0 & 0 - 0 + 0 \\ \sin x \cos y + \cos x \sin y + 0 & -\sin x \sin y + \cos x \cos y + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= F(X + Y)$$

Hence proved.

14. Show that

$$(i) \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$(i) \text{ L.H.S.: } \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Multiply both the matrices

$$\begin{bmatrix} 5(2)+(-1)3 & 5(1)+(-1)4 \\ 6(2)+7(3) & 6(1)+7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$$

$$\text{R.H.S.: } \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

Multiply both the matrices

$$= \begin{bmatrix} 2(5)+1(6) & 2(-1)+1(7) \\ 3(5)+4(6) & 3(-1)+4(7) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$$

L.H.S. \neq R.H.S.

$$(ii) \text{ L.H.S.: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Multiply both the matrices

$$= \begin{bmatrix} 1(-1)+2(0)+3(2) & 1(1)+2(-1)+3(3) & 1(0)+2(1)+3(4) \\ 0(-1)+1(0)+0(2) & 0(1)+1(-1)+0(3) & 0(0)+1(1)+0(4) \\ 1(-1)+1(0)+0(2) & 1(1)+1(-1)+0(3) & 1(0)+1(1)+0(4) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

R.H.S.:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1(1)+1(0)+0(1) & (-1)2+1(1)+0(1) & (-1)3+1(0)+0(0) \\ 0(1)+(-1)0+1(1) & (0)2+1(-1)+1(1) & (0)3+0(-1)+1(0) \\ 2(1)+3(0)+4(1) & 2(2)+3(1)+4(1) & 2(3)+3(0)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix}$$

L.H.S. \neq R.H.S.

15. Find $A^2 - 5A + 6I$, if A is

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution:

Find $A^2 = A \times A$

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Find $5A$

$$5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

Find $6I$

$$6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$\begin{aligned} A^2 - 5A + 6I &= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix} \end{aligned}$$

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 \equiv 6A^2 + 7A + 2I = 0$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$6A^2 = 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Now,

$$\begin{aligned} A^3 - 6A^2 + 7A + 2I &= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Which is a zero matrix.

Therefore, $A^3 - 6A^2 + 7A + 2I = 0$ (Proved)

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$.

Solution:

$$A^2 = kA - 2I$$

$$\begin{aligned} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} &= k \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} &= \begin{bmatrix} 3k-2 & -2k-0 \\ 4k-0 & -2k-2 \end{bmatrix} \end{aligned}$$

Equate all the corresponding values to find the value of k ,

$$1 = 3k - 2 \Rightarrow k = 1$$

$$-2 = -2k \Rightarrow k = 1$$

$$4 = 4k \Rightarrow k = 1$$

$$-4 = -2k - 2 \Rightarrow k = 1$$

The value of k is 1.

18. If $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Solution:

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

Now,
R.H.S. =

$$(I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha + \sin \alpha \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \tan \frac{\alpha}{2} \\ -\cos \alpha \tan \frac{\alpha}{2} + \sin \alpha & \sin \alpha \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos \alpha \cos \frac{\alpha}{2} + \sin \alpha \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} & \frac{-\sin \alpha \cos \frac{\alpha}{2} + \cos \alpha \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \\ \frac{-\cos \alpha \sin \frac{\alpha}{2} + \sin \alpha \cos \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} & \frac{\sin \alpha \sin \frac{\alpha}{2} + \cos \alpha \cos \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \frac{-\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \\ \frac{\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \frac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \frac{-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

L.H.S.

Hence proved.

19. A trust fund has Rs. 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs.30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

- (a) Rs. 1800 (b) Rs. 2000

Solution:

Let x be the investment in first bond, then the investment in the second bond will be Rs. $(30,000 - x)$.

Interest paid by first bond is 5% per year and interest paid by second bond is 7% per year.

Matrix of investment $[x \quad 30000-x]$

Matrix of annual interest per year is

$$\left[\begin{array}{c} 5 \\ 100 \\ 7 \\ 100 \end{array} \right]$$

To obtain an annual total interest of Rs. 1800, we have

$$[x \quad 30000-x] \left[\begin{array}{c} 5 \\ 100 \\ 7 \\ 100 \end{array} \right] = 1800$$

$$\left[\frac{5x}{100} + \frac{7(30000-x)}{100} \right] = 1800$$

$$\frac{210000 - 2x}{100} = 1800$$

$$210000 - 2x = 180000$$

$$x = 15000$$

The investment in first bond is Rs. 15,000

And investment in second bond is Rs. $(30000 - 15000) = \text{Rs. } 15,000$

To obtain an annual total interest of Rs. 2000, we have

$$[x \quad 30000-x] \left[\begin{array}{c} 5 \\ 100 \\ 7 \\ 100 \end{array} \right] = 2000$$

$$\frac{210000 - 2x}{100} = 2000$$

$$\text{or } x = \text{Rs. } 5000$$

The investment in first bond is Rs. 5,000

And investment in second bond is Rs. $(30000 - 5000) = \text{Rs. } 25,000$

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra. Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

Solution:

Let the selling prices of each book as a 3×1 matrix

$$\begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

Total amount received by selling all books

$$12[10 \ 8 \ 10] \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} = [120 \ 96 \ 120] \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= 9600 + 5760 + 4800 \\ = 20160$$

Total amount received by selling all the books is Rs. 20160.

21. The restriction on n , k and p so that $PY + WY$ will be defined are:

- (A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$
 (C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

Solution:

Option (A) is correct.

$PY + WY = P(\text{order of matrix, } p \times k) \times Y(\text{order of matrix, } 3 \times k) + W(\text{order of matrix, } n \times k)$
 $Y(\text{order of matrix, } 3 \times k)$

Here $k = 3$ and $p = n$

22. If $n = p$, then the order of the matrix $7X - 5Z$ is:

- (A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

Solution: Option (B) is correct.

The order of matrices X and Z are equal, since $n = p$

The order of $7X - 5Z$ is same as the order of X and Z .

The order of $7X - 5Z$ is either $2 \times n$ or $2 \times p$. (Given $n = p$)